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Symmetries and invariant solutions for the geometric heat flows

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Abstract

We study Lie symmetries and invariant solutions of the geometric heat flows. The basic similarity reductions for the GHE are performed. Reduced equations and exact solutions associated with the symmetries are obtained. Group-invariant solutions and reductions for the affine case are also discussed in a special case.

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1. Introduction

In the past twenty years, there has been much research devoted to the study of evolutions of plane curves

$$\mathcal{C}_t = k\mathcal{N}, \tag{1}$$

where $\mathcal{N}$ and k are, respectively, a choice of unit (inward) normal for $\mathcal{C}$ and the curvature with respect to $\mathcal{N}$. This evolution appears in a number of different pure and applied areas such as differential geometry, crystal growth, image processing, computer vision and physics, etc, see [1–14] and references therein for a more extensive discussion of the many properties associated with this flow.

The flow is referred to as Euclidean curve shortening flow, in the sense that the Euclidean perimeter shrinks when the curve evolves according to equation (1) [4, 15–17]. The behavior of an embedded curve evolving according to this flow has been well studied. Gage and Hamilton have proved that a convex embedded curve converges to a round point under this evolution [3, 15]. Grayson [17] has shown that a nonconvex embedded curve converges to a convex one, and from there to a round point according to the Gage and Hamilton result. This equation was also called the geometric heat equation (GHE). This flow has a number of nice properties which make it very useful in morphological image processing, and in particular the basis of a nonlinear scale-space invariant to rotations and translations for shape representation [9, 18]. A related flow, based upon the affine geometry of the curve, is given by

$$\mathcal{C}_t = k^{\frac{1}{3}}\mathcal{N}, \tag{2}$$

which is called the affine geometric heat equation. This flow shares many of the same properties with the curve shortening flow but gives rise to a more general affine invariant multiscale space [18–20]. More discussions on (2) can be found in [21–23] and references therein. Locally (2) may be written as

$$u_t = u_{xx}^{\frac{1}{3}},$$

whose Lie symmetries and group-invariant solutions were discussed in detail in [23].

In the level set method [24, 25], the parameterized curve $\mathcal{C}(p, t)$ is embedded into a surface, which is called the level set function $u(x, y, t) : \mathbb{R}^2 \times [0, T] \to \mathbb{R}$. The curve $\mathcal{C}$ is the zero-level set of this function $u(x, y, t)$:

$$\mathcal{C} = \{(x, y) : u(x, y, t) = 0\}.$$

The evolution equation for u is derived from the constraint that at any time t we should have

$$u(\mathcal{C}, t) = u(\mathcal{X}(t), \mathcal{Y}(t), t) = 0, \tag{3}$$

and differentiating (3) with respect to t we obtain

$$u_t + \nabla u \cdot \mathcal{C}_t = 0. \tag{4}$$

Substituting the general form of the curve evolution equation (1), which depends on local geometry of the curve, into (4) above yields

$$u_t + \nabla u \cdot k\mathcal{N} = 0.$$

Note that for the zero level, the following relation $\mathcal{N} = -\nabla u/\|\nabla u\|$ holds, then an evolution equation for u is given by

$$u_t = k\|\nabla u\|, \tag{5}$$

where

$$k = \nabla \cdot \left(\frac{\nabla u}{\|\nabla u\|}\right) = \frac{u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{\left(u_x^2 + u_y^2\right)^{3/2}},$$

which is in fact the curvature of the curve $\mathcal{C}$ regarded as the level set of the corresponding evolution [18, 24, 26]. This allows us to rewrite equation (5) completely in terms of u and its derivatives as

$$u_t = \frac{u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{u_x^2 + u_y^2}. \tag{6}$$

This flow is also referred to as the geometric heat equation since it is a result of applying the previous geometric heat equation (1) to the zero-level curve of the level set function u.

Similarly, the affine invariant heat flow (2) in terms of the level set function u can be written as

$$u_t = \left(u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}\right)^{\frac{1}{3}}. \tag{7}$$

It is well known that exact solutions play a crucial role in the study of asymptotic behavior, blow up or extinction and geometric properties of invariant geometric flows. For instance, it was shown that when a locally convex closed immersed curve collapses into a point, its asymptotic shape must be one of the contracting self-similar solutions of (1) classified in [22, 27, 28]. A ‘grim reaper’, a travelling wave solution first observed in [3] has been used to describe the asymptotic profile of ‘type-II singularity’ of curves [22, 23]. A contracting spiral wave solution was also used in the analysis of singularities of curves [2, 22]. The purpose of this paper is to discuss symmetries and solutions of GHE (6) and affine GHE (7).

The outline of this paper is as follows. In section 2, we derive the Lie symmetry group of GHE (6). It is reduced to two-dimensional PDEs when the arbitrary functions of its infinitesimal transformations are confined to arbitrary constants in section 3. We provide symmetry group analysis for (10) and (11) in sections 4 and 5, respectively. And in section 6, reduced ODEs and group-invariant solutions of GHE are presented. Exact solutions of affine GHE are obtained for a special case in section 7. Section 8 contains a concluding remark on this work.

2. Lie symmetry of the geometric heat flow

The classical method for finding symmetry reductions of PDE is the Lie group method of infinitesimal transformations. To apply the classical method to (6), we consider the one-parameter Lie group of infinitesimal transformations in (x, y, t, u) given by

$$\begin{aligned}
x^* &= x + \epsilon\xi_1(x, y, t, u) + O(\epsilon^2),\\
y^* &= y + \epsilon\xi_2(x, y, t, u) + O(\epsilon^2),\\
t^* &= t + \epsilon\xi_3(x, y, t, u) + O(\epsilon^2),\\
u^* &= u + \epsilon\xi_4(x, y, t, u) + O(\epsilon^2),
\end{aligned}$$

where ϵ is the group parameter. One requires that this transformation leaves the set

$$S_\Delta = \{u(x, y, t)|\Delta = 0\}$$

invariant, where $\Delta[u] \equiv u_t - \left(u_y^2u_{xx} - 2u_xu_yu_{xy} + u_x^2u_{yy}\right)/\left(u_x^2 + u_y^2\right)$. This yields an overdetermined, linear system of equations for the infinitesimals $\xi_1(x, y, t, u)$, $\xi_2(x, y, t, u)$, $\xi_3(x, y, t, u)$ and $\xi_4(x, y, t, u)$. The associated Lie algebra is realized by vector fields of the form

$$X = \xi_1(x, y, t, u)\frac{\partial}{\partial x} + \xi_2(x, y, t, u)\frac{\partial}{\partial y} + \xi_3(x, y, t, u)\frac{\partial}{\partial t} + \xi_4(x, y, t, u)\frac{\partial}{\partial u}. \tag{8}$$

The set S_Δ is invariant under the transformation (8) provided that $\mathrm{pr}^{(2)}X(\Delta)|_{\Delta\equiv 0} = 0$ where $\mathrm{pr}^{(2)}X$ is the second prolongation of the vector field (8), which is given explicitly in terms of ξ_1, ξ_2, ξ_3 and ξ_4 [29–31]. This procedure yields an overdetermined system. Solving it gives Lie symmetries of (6)

$$\begin{aligned}
\xi_1 &= F_2(u)x - F_4(u)y + F_5(u),\\
\xi_2 &= F_4(u)x + F_2(u)y + F_1(u),\\
\xi_3 &= 2F_2(u)t + F_3(u),\\
\xi_4 &= F_6(u),
\end{aligned}$$

where $F_i(u) (i = 1, \ldots, 6)$ are the arbitrary functions of u. Therefore, the symmetry group of equation (6) is spanned by the vector fields

$$\begin{aligned}
&F_5(u)\frac{\partial}{\partial x}, \quad F_1(u)\frac{\partial}{\partial y}, \quad F_3(u)\frac{\partial}{\partial t}, \quad F_6(u)\frac{\partial}{\partial u}, && \text{(gauge translation)},\\
&F_2(u)x\frac{\partial}{\partial x} + F_2(u)y\frac{\partial}{\partial y} + 2F_2(u)t\frac{\partial}{\partial t}, && \text{(gauge scaling)},\\
&-F_4(u)y\frac{\partial}{\partial x} + F_4(u)x\frac{\partial}{\partial y}, && \text{(gauge rotation)}.
\end{aligned}$$

It is interesting to note that if u is a solution of (6), so is $f(u)$ for any arbitrary differentiable functions f.

In the following, we confine $F_i(u)(i = 1, \ldots, 5)$ to constants $k_i(i = 1, \ldots, 5)$ and set $k_6 = 1$, then

$$\xi_1^* = k_2 x - k_4 y + k_5,$$
$$\xi_2^* = k_4 x + k_2 y + k_1,$$
$$\xi_3^* = 2k_2 t + k_3,$$
$$\xi_4^* = 1.$$

Having determined the infinitesimals, the symmetry variables are found by solving the characteristic equation

$$\frac{\mathrm{d}x}{\xi_1^*} = \frac{\mathrm{d}y}{\xi_2^*} = \frac{\mathrm{d}t}{\xi_3^*} = \frac{\mathrm{d}u}{\xi_4^*} \tag{9}$$

or the corresponding invariant-surface condition

$$\Psi \equiv \xi_1^* u_x + \xi_2^* u_y + \xi_3^* u_t - \xi_4^* = 0.$$

3. Reduction of the geometric heat flow to two-dimensional PDEs

There are four independent reductions that are given as follows:

Case 1. $k_4 \neq 0, k_2 \neq 0$. Integration of (9) gives the reduced variables

$$\xi = \mathrm{e}^{-k_2 u}((x - x_0)^2 + (y - y_0)^2), \qquad \eta = \mathrm{e}^{-k_2 u}(t - t_0)$$

in which

$$x_0 = -\frac{k_2 k_5 + k_1 k_4}{k_4^2 + k_2^2}, \qquad y_0 = \frac{k_4 k_5 - k_1 k_2}{k_4^2 + k_2^2}, \qquad t_0 = -\frac{k_3}{2k_2}$$

and the following reduction for the fields

$$k_4 u - \arctan\frac{y - y_0}{x - x_0} = v(\xi, \eta).$$

Substitution of the two reduction ansatz into (6) gives

$$\left(4\xi^2 v_\xi^2 + 1\right) v_\eta = 2\left(2\xi v_{\xi\xi} + 4\xi^2 v_\xi^3 + 3v_\xi\right). \tag{10}$$

Case 2. $k_4 \neq 0, k_2 = 0$. Integration of (9) yields the reduced variables

$$\xi = (x - x_0)^2 + (y - y_0)^2, \qquad \eta = t - k_3 u$$

where $x_0 = -k_1/k_4$, $y_0 = k_5/k_4$, and the reduction for the fields is exactly the same as in case 1. By the substitution of the reduction ansatz in (6), we obtain equation (10).

Case 3. $k_4 = 0, k_2 \neq 0$. Integration of (9) provides the following reduction:

$$\xi = \mathrm{e}^{-k_2 u}(x - x_0), \qquad \eta = \mathrm{e}^{-k_2 u}(y - y_0)$$

and

$$\mathrm{e}^{-k_2 u}(t - t_0) = v(\xi, \eta)$$

where $x_0 = -k_5/k_2$, $y_0 = -k_1/k_2$ and $t_0 = -k_3/(2k_2)$. Substitution of the two reduction ansatz above into (6) gives

$$-1 = \frac{v_\eta^2 v_{\xi\xi} - 2v_\xi v_\eta v_{\xi\eta} + v_\xi^2 v_{\eta\eta}}{v_\xi^2 + v_\eta^2}. \tag{11}$$

Case 4. $k_4 = 0, k_2 = 0, k_1^2 + k_3^2 + k_5^2 \neq 0$. Integration of (9) yields the following reduction:

$$\xi = x - k_5 u, \qquad \eta = y - k_1 u$$

and the reduction for the field

$$t - k_3 u = v(\xi, \eta),$$

where v satisfies (11).

As explained before, we need the group-invariant solutions of (10) and (11) in order to construct the solutions of GHE. In the following two sections, we shall further reduce (10) and (11) by using their symmetries.

4. Symmetry group analysis for (10)

As is well known, the Lie group theoretic method plays an important role in finding exact solutions and performing symmetry reductions of differential equations. Since any linear combination of infinitesimal generators is also an infinitesimal generator, there are always infinitely many different symmetry subgroups for the differential equation. So, a mean of determining which subgroups would give essentially different types of solutions is necessary and significant for a complete understanding of the invariant solutions. As any transformation in the full symmetry group maps a solution to another solution, it is sufficient to find invariant solutions which are not related by transformations in the full symmetry group, this has led to the concept of an optimal system [29–31]. The problem of finding an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. For one-dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation. This problem is attacked by the naive approach of taking a general element in the Lie algebra and subjecting it to various adjoint transformations so as to simplify it as much as possible. The idea of using the adjoint representation to classify group-invariant solutions was due to Ovsiannikov [31].

The Lie algebra of infinitesimal symmetries of (10) is spanned by the following five vector fields:

$$\begin{aligned}
X_1 &= 2\sqrt{\xi}\cos v\frac{\partial}{\partial \xi} - \frac{\sin v}{\sqrt{\xi}}\frac{\partial}{\partial v},\\
X_2 &= 2\sqrt{\xi}\sin v\frac{\partial}{\partial \xi} + \frac{\cos v}{\sqrt{\xi}}\frac{\partial}{\partial v},\\
X_3 &= \frac{\partial}{\partial v},\\
X_4 &= \xi\frac{\partial}{\partial \xi} + \eta\frac{\partial}{\partial \eta},\\
X_5 &= \frac{\partial}{\partial \eta}.
\end{aligned} \tag{12}$$

The commutation relations of this Lie algebra are presented in table 1, where the (i, j)th entry represents the commutator $[X_i, X_j]$.

The adjoint action is given by the Lie series

$$\mathrm{Ad}(\exp(\epsilon X_i)X_j) = X_j - \epsilon[X_i, X_j] + \frac{\epsilon^2}{2}[X_i, [X_i, X_j]] - \cdots,$$

where $[X_i, X_j]$ is the commutator for the Lie algebra and ϵ is a parameter. We can write the adjoint action for the Lie algebra (12). It is listed in table 2, where the (i, j)th entry gives $\mathrm{Ad}(\exp(\epsilon X_i)X_j)$.

Table 1. Composition table for (12).

	X_1	X_2	X_3	X_4	X_5
X_1	0	0	X_2	$\frac{1}{2}X_1$	0
X_2	0	0	$-X_1$	$\frac{1}{2}X_2$	0
X_3	$-X_2$	X_1	0	0	0
X_4	$-\frac{1}{2}X_1$	$-\frac{1}{2}X_2$	0	0	$-X_5$
X_5	0	0	0	X_5	0

Table 2. The adjoint representation of (12).

Ad $(\epsilon\cdot)$	X_1	X_2	X_3	X_4	X_5
X_1	X_1	X_2	$X_3-\epsilon X_2$	$X_4-\frac{\epsilon}{2}X_1$	X_5
X_2	X_1	X_2	$X_3+\epsilon X_1$	$X_4-\frac{\epsilon}{2}X_2$	X_5
X_3	$X_1\cos\epsilon+X_2\sin\epsilon$	$X_2\cos\epsilon-X_1\sin\epsilon$	X_3	X_4	X_5
X_4	$e^{\frac{\epsilon}{2}}X_1$	$e^{\frac{\epsilon}{2}}X_2$	X_3	X_4	$e^{\epsilon}X_5$
X_5	X_1	X_2	X_3	$X_4-\epsilon X_5$	X_5

Theorem 1. *A one-dimensional optimal system of (12) is given by*

$$\begin{aligned} &W_1=X_4, \qquad W_2=X_4+\alpha X_3 \quad (\alpha\neq 0), \qquad W_3=X_3, \qquad W_4=X_3+X_5,\\ &W_5=X_3-X_5, \qquad W_6=X_1, \qquad W_7=X_5, \qquad W_8=X_5+X_1. \end{aligned} \tag{13}$$

Proof. Let $X=\sum_{i=1}^{5}a_iX_i$. First of all, using $\mathrm{Ad}\exp(\epsilon X_3)$, we may rotate X_1 and X_2. As a result, we shall always assume that $a_2=0$ in the following discussion. Now we claim that the space spanned by a nonzero X must be equivalent to some W_i. We consider three cases separately.

Case 1. If $a_4\neq 0$, scaling X if necessary, we can assume that $a_4=1$. So X is equivalent to

$$X=X_4+a_1X_1+a_3X_3+a_5X_5.$$

Acting on this vector by $\mathrm{Ad}\exp(a_5X_5)$, we can make the coefficient of X_5 vanish. And X is reduced to

$$X=X_4+a_1X_1+a_3X_3.$$

Applying $\mathrm{Ad}\exp(\epsilon_1X_1)$ and $\mathrm{Ad}\exp(\epsilon_2X_2)$ to this X, we obtain a new vector

$$X=X_4+\tilde{a}_1X_1+\tilde{a}_2X_2+a_3X_3,$$

where $\tilde{a}_1=a_1+a_3\epsilon_2-\epsilon_1/2$ and $\tilde{a}_2=-a_3\epsilon_1-\epsilon_2/2$, and they vanish after choosing

$$\epsilon_1=\frac{2a_1}{1+4a_3^2}, \qquad \epsilon_2=-\frac{4a_1a_3}{1+4a_3^2}.$$

Thus X is equivalent to one of the following vector fields X_4 and $X_4+\alpha X_3(\alpha\neq 0)$.

Case 2. If $a_4=0$ and $a_3\neq 0$, we scale to make $a_3=1$. Use $\mathrm{Ad}\exp(-a_1X_1)$ to eliminate a_1. After acted by $\mathrm{Ad}\exp(\epsilon X_5)$ for suitable ϵ, we obtain three inequivalent generators X_3, X_3+X_5 and X_3-X_5.

Case 3. If $a_4=0$ and $a_3=0$, in this case, X is simplified to $X=a_1X_1+a_5X_5$.

If $a_5 \neq 0$, we take $a_5 = 1$. After using the adjoint action of the group generated by X_4, we conclude that X is equivalent to X_5, $X_5 + X_1$ and $X_5 - X_1$. Acting on $X_5 - X_1$ by $\mathrm{Ad}\exp(\pi X_3)$, we obtain $X_5 + X_1$. Thus any one-dimensional subalgebra spanned by X is equivalent to one spanned by either X_5 or $X_5 + X_1$.

If $a_5 = 0$, then the only remaining vectors are the multiples of X_1, on which the adjoint representation acts trivially. Thus X is reduced to X_1. □

Thus, we have shown that any one-dimensional subspace of (12) is equivalent to that of the subspaces spanned by $W_1, \ldots, W_8$. It remains to prove that any two one-dimensional subalgebras obtained above are mutually inequivalent [23, 32]. We shall accomplish this by introducing some adjoint invariants. Recall that a real function ϕ on a Lie algebra $\mathfrak{g}$ is called an invariant if $\phi(\mathrm{Ad}(\mathfrak{g})X) = \phi(X)$ for all $X \in \mathfrak{g}$ and $\mathfrak{g}$ in the corresponding Lie group G. For two vectors X and Y, generate conjugate one-dimensional subalgebra, it is necessary that $\phi(X) = \phi(Y)$ for any invariant ϕ. Let $X = \sum_{i=1}^{5} a_i X_i$ be a general vector for (12), then ϕ can be regarded as a function of $a_1, \ldots, a_5$.

Lemma 2. *$A = a_3$, $B = a_4$ are invariants.*

Proof. This can be easily seen from table 2. □

Lemma 3. *The following function is an invariant:*

$$C = \operatorname{sign} a_5.$$

Proof. Since the actions of $\mathrm{Ad}\exp(\epsilon X_i)$, $i \neq 4$, do not change the values of a_5, it is sufficient to check the invariance of C under the action of $\mathrm{Ad}\exp(\epsilon X_4)$. We denote the new coefficient by $\tilde{a}_5$. Under $\mathrm{Ad}\exp(\epsilon X_4)$, $\tilde{a}_5 = \mathrm{e}^{\epsilon} a_5$, thus C is an invariant. □

Lemma 4. *D is an invariant, where*

$$D = \begin{cases} 1 & a_3 = a_4 = 0,\ a_1^2 + a_2^2 \neq 0, \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Since a_3 and a_4 are invariants, it suffices to check the invariance of D under $a_3 = a_4 = 0$. However, observe that $\mathrm{Ad}\exp(\epsilon X_i)$, $i = 1, 2, 5$, do not change X_1 and X_2. We only need to check the action of $\mathrm{Ad}\exp(\epsilon_1 X_3)$ and $\mathrm{Ad}\exp(\epsilon_2 X_4)$. We denote the new coefficients by $\tilde{a}_1$ and $\tilde{a}_2$.

In fact, after acted by $\mathrm{Ad}\exp(\epsilon_1 X_3)$, $\tilde{a}_1$ and $\tilde{a}_2$ satisfy $\tilde{a}_1^2 + \tilde{a}_2^2 = a_1^2 + a_2^2$, and then D is unchanged. On the other hand, under $\mathrm{Ad}\exp(\epsilon_2 X_4)$, $\tilde{a}_1$ and $\tilde{a}_2$ satisfy $\tilde{a}_1^2 + \tilde{a}_2^2 = \mathrm{e}^{\epsilon_2}\left(a_1^2 + a_2^2\right)$. Hence D is also unchanged.

We conclude that D is actually an invariant. □

Now, we claim that different W_i's are mutually inequivalent. We evaluate all invariants for each and put the results in table 3. It is clear from table 3 that for different i, or the same i but with different parameters, they are inequivalent. We have established the optimality of the system. So theorem 1 holds.

We have obtained eight inequivalent one-dimensional subalgebras. Each subalgebra will provide a reduction to an ODE. We shall consider one of the subalgebras, i.e., $W_6 = X_1$ in some details, as an example. The results for the other one-dimensional subalgebras can be obtained in a similar manner.

Table 3. Invariants for (13).

	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8
A	0	α	1	1	1	0	0	0
B	1	1	0	0	0	0	0	0
C	0	0	0	1	-1	0	1	1
D	0	0	0	0	0	1	0	1

For W_6, the characteristic equation is

$$\frac{\mathrm{d}\xi}{2\sqrt{\xi}\cos v} = \frac{\mathrm{d}\eta}{0} = \frac{\mathrm{d}v}{-\frac{\sin v}{\sqrt{\xi}}}.$$

Global invariants of this group are

$$z = \eta, \qquad \lambda = \xi \sin^2 v,$$

so that a group-invariant solution $\lambda = g(z)$ takes the form

$$\xi \sin^2 v = g(z).$$

Solving for the derivatives of v with respect to ξ, η in terms of those of λ with respect to z and substituting these expressions into (10), we find the reduced ODE

$$g' = 0,$$

where and hereafter the primes denote differentiation with respect to z. It is solved by

$$g(z) = C,$$

where C is a nonzero arbitrary constant.

Then we obtain an exact solution of (10) with

$$v(\xi, \eta) = \arcsin \frac{C_1}{\sqrt{\xi}},$$

where C_1 is a nonzero arbitrary constant.

Not all groups will generate group-invariant solutions. The criterion for the existence of such solutions can be found in [31]. However, it is not necessary to examine for any case. One simply discovers during the derivation of the similarity variables that the desired reduction in the number of independent variables does not occur. Algebra W_3 fails this test and provides no group-invariant solutions. And all other algebras generate reductions of (10) to ODEs. We run through the individual subalgebras and obtain the reduction formula and the corresponding invariant equations written in terms of the invariants. The results for the other one-dimensional subalgebras are listed in table A1. In the reduced equations, we always take the second invariant as a function of the first invariant. Note that it remains necessary to solve these ODEs to obtain the group-invariant solutions explicitly, and in most cases this is still very difficult. Once the reduced equation in column 5 is solved, the corresponding relation in column 4 explicitly defines a surface in (ξ, η, v)-space.

Note that all second-order ODEs in table A1 can be reduced to first-order ODEs easily. The reduced ODEs for W_1 and W_7 are solved here for particular solutions which then provide complete analytic solutions of (10).

Consider the reduced ODE for W_1:

$$4zg'' + 4z^2(z+2)g'^3 + (z+6)g' = 0.$$

Table 4. Composition table for (14).

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	0	0	0	Y_2	Y_1
Y_2	0	0	0	$-Y_1$	Y_2
Y_3	0	0	0	0	$2Y_3$
Y_4	$-Y_2$	Y_1	0	0	0
Y_5	$-Y_1$	$-Y_2$	$-2Y_3$	0	0

Introduce a new function $h(z)$ which satisfies $h(z) = g'(z)$, then the equation above can be reduced to

$$4zh' + 4z^2(z+2)h^3 + (z+6)h = 0,$$

and yields

$$h(z) = \pm\frac{1}{z\sqrt{C_1 z\,\mathrm{e}^{\frac{z}{2}} - 4}}.$$

Thus,

$$v(\xi,\eta) = \pm\int^{\frac{\xi}{\eta}} \frac{1}{s\sqrt{C_1 s\,\mathrm{e}^{\frac{s}{2}} - 4}}\,\mathrm{d}s + C_2$$

is a group-invariant solution of (10) corresponding to W_1.

Similar to the above analysis, we obtain an exact solution associated with W_7 given by

$$v(\xi,\eta) = \pm\arctan\sqrt{-1 + C_1\xi} + C_2.$$

Many more solutions are certainly possible and can be obtained through the solutions of the reduced ODEs.

5. Symmetry group analysis for (11)

Similar to the previous section, symmetry group analysis for (11) is accomplished in this section. First, we shall determine the symmetry group of (11), classify one-parameter subgroups up to the adjoint representation and finally obtain the reduced ODEs or some group-invariant solutions for the one-dimensional optimal systems.

Theorem 5. *The Lie algebra of infinitesimal symmetries of (11) is spanned by the following five vector fields:*

$$\begin{aligned} &Y_1 = \frac{\partial}{\partial\xi}, \qquad Y_2 = \frac{\partial}{\partial\eta}, \qquad Y_3 = \frac{\partial}{\partial v}, \\ &Y_4 = -\eta\frac{\partial}{\partial\xi} + \xi\frac{\partial}{\partial\eta}, \qquad Y_5 = \xi\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta} + 2v\frac{\partial}{\partial v}. \end{aligned} \tag{14}$$

The commutation relation and the action of the adjoint representation for the Lie algebra (14) can be found in tables 4 and 5, respectively.

Let us use the notation

$$\begin{aligned} &V_1 = Y_4, \qquad V_2 = Y_4 + Y_3, \qquad V_3 = Y_4 - Y_3, \qquad V_4 = Y_4 + \alpha Y_5 \quad (\alpha \neq 0), \\ &V_5 = Y_5, \qquad V_6 = Y_1, \qquad V_7 = Y_3, \qquad V_8 = Y_1 + Y_3. \end{aligned} \tag{15}$$

Table 5. The adjoint representation of (14).

$\mathrm{Ad}(\epsilon\cdot)$	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	Y_1	Y_2	Y_3	$Y_4-\epsilon Y_2$	$Y_5-\epsilon Y_1$
Y_2	Y_1	Y_2	Y_3	$Y_4+\epsilon Y_1$	$Y_5-\epsilon Y_2$
Y_3	Y_1	Y_2	Y_3	Y_4	$Y_5-2\epsilon Y_3$
Y_4	$Y_1\cos\epsilon+Y_2\sin\epsilon$	$Y_2\cos\epsilon-Y_1\sin\epsilon$	Y_3	Y_4	Y_5
Y_5	$\mathrm{e}^{\epsilon}Y_1$	$\mathrm{e}^{\epsilon}Y_2$	$\mathrm{e}^{2\epsilon}Y_3$	Y_4	Y_5

Table 6. Invariants for (15).

	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
E	1	1	1	1	0	0	0	0
F	0	0	0	α	1	0	0	0
H	0	1	−1	0	0	0	1	1
P	0	0	0	0	0	1	0	1

Theorem 6. *The vectors $V_1, \ldots, V_8$ form an optimal system of one-dimensional subalgebra for (14).*

Let $Y = \sum_{i=1}^{5} b_i Y_i$ be a general vector for (14). Similarly to the proof of theorem 1, it is easy to show that each one-dimensional subalgebra of (14) is equivalent to one member in V_i, $(i = 1, \ldots, 8)$. Now we claim that they are inequivalent and hence form an optimal system. To prove this, we define some adjoint invariants.

Lemma 7. *$E = b_4$, $F = b_5$ are invariants.*

Lemma 8. *Define*

$$H = \begin{cases} \operatorname{sign} b_3, & b_5 = 0, \\ 0, & \text{otherwise}. \end{cases}$$

Then H is an invariant.

Proof. Since b_5 is an invariant, it suffices to check the invariance of H under $b_5 = 0$. Note that $\mathrm{Ad}\exp(\epsilon Y_i)$, $i \neq 5$, do not change the value of b_3. We only need to check the action of $\mathrm{Ad}\exp(\epsilon Y_5)$. In fact, after acted by $\mathrm{Ad}\exp(\epsilon Y_5)$, the new coefficients of Y_3, say $\tilde{b}_3$, satisfy $\tilde{b}_3 = \mathrm{e}^{2\epsilon} b_3$, and then H is unchanged. □

Lemma 9. *The following function is an invariant:*

$$P = \begin{cases} 1, & b_4 = b_5 = 0, b_1^2 + b_2^2 \neq 0, \\ 0, & \text{otherwise}. \end{cases}$$

The proof is similar to that of lemma 4.

Now evaluate all invariants at each $V_i (i = 1, \ldots, 8)$ and put the results in table 6. It is clear from table 6 that for different i, or the same i but with different parameters, they are inequivalent. Then theorem 6 holds.

We run through the individual subalgebras (15) and obtain the reduction formula and the corresponding invariant equations written in terms of the invariants. Note that V_6 and V_7 cannot yield group-invariant solutions. All other group reductions are presented in table A2. As explained before, in the reduced equations we always take the second invariant as a function of the first invariant.

Table 7. The adjoint representation of (18).

Ad($\epsilon\cdot$)	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	Z_7
Z_1	Z_1	Z_2	Z_3	$Z_4-\epsilon Z_2$	Z_5	$Z_6-\epsilon Z_1$	$Z_7-3\epsilon Z_1$
Z_2	Z_1	Z_2	Z_3	Z_4	$Z_5-\epsilon Z_1$	$Z_6+\epsilon Z_2$	Z_7
Z_3	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	$Z_7-2\epsilon Z_3$
Z_4	$Z_1+\epsilon Z_2$	Z_2	Z_3	Z_4	$Z_5+\epsilon Z_6-\epsilon^2 Z_4$	$Z_6+2\epsilon Z_4$	$Z_7+3\epsilon Z_4$
Z_5	Z_1	$Z_2+\epsilon Z_1$	Z_3	$Z_4+\epsilon Z_6-\epsilon^2 Z_5$	Z_5	$Z_6-2\epsilon Z_5$	$Z_7-3\epsilon Z_5$
Z_6	$e^{\epsilon} Z_1$	$e^{-\epsilon} Z_2$	Z_3	$e^{-2\epsilon} Z_4$	$e^{2\epsilon} Z_5$	Z_6	Z_7
Z_7	$e^{3\epsilon} Z_1$	Z_2	$e^{2\epsilon} Z_3$	$e^{-3\epsilon} Z_4$	$e^{3\epsilon} Z_5$	Z_6	Z_7

Except for the reduced ODE for V_5, all other ODEs can be reduced to first-order ODEs. Once they are solved, exact solutions for (11) can be obtained. Here, we only write the group-invariant solution corresponding to V_1:

$$v(\xi,\eta) = -\tfrac{1}{2}(\xi^2+\eta^2)+C_1,$$

and the exact solution corresponding to V_8:

$$v(\xi,\eta) = -\tfrac{1}{2}\ln(1+\tan^2(\eta+C_1))+C_2,$$

where and hereafter C_1 and C_2 denote arbitrary constants.

6. Group-invariant solutions for the geometric heat flow

Since we have reduced the geometric heat flow for surface to (10) and (11) in section 3, the further symmetry analysis for the two PDEs is accomplished in sections 4 and 5. Combining the results and conclusions in sections 3–5 together, we can obtain the reduced equations, or group-invariant solutions, for the geometric heat equation.

Case 1. $k_4\neq 0, k_2\neq 0$. The group-invariant solutions for the GHE are given by

$$k_4 u - \arctan\frac{y-y_0}{x-x_0} = v(e^{-k_2 u}((x-x_0)^2+(y-y_0)^2), e^{-k_2 u}(t-t_0)),$$

where

$$x_0 = -\frac{k_2k_5+k_1k_4}{k_4^2+k_2^2}, \qquad y_0 = k_4k_5-k_1k_2k_4^2+k_2^2, \qquad t_0=-\frac{k_3}{2k_2}.$$

Case 2. $k_4\neq 0, k_2=0$. In this case, the group-invariant solutions for the GHE should satisfy

$$k_4 u - \arctan\frac{y-\frac{k_5}{k_4}}{x+\frac{k_1}{k_4}} = v\left(\left(x+\frac{k_1}{k_4}\right)^2+\left(y-\frac{k_5}{k_4}\right)^2, t-k_3u\right).$$

Case 3. $k_4=0, k_2\neq 0$. The group-invariant solutions of the GHE are given implicitly by

$$e^{-k_2u}\left(t+\frac{k_3}{2k_2}\right) = v\left(e^{-k_2u}\left(x+\frac{k_5}{k_2}\right), e^{-k_2u}\left(y+\frac{k_1}{k_2}\right)\right).$$

Case 4. $k_4=0, k_2=0, k_1^2+k_3^2+k_5^2\neq 0$. In this case, the group-invariant solutions for the GHE can be expressed as

$$t-k_3u = v(x-k_5u, y-k_1u).$$

In the above four cases, the function v, or the equations it satisfies, can be found in table A1 for cases 1 and 2, and in table A2 for cases 3 and 4.

Here, we only present subsequently three group-invariant solutions of the GHE for illustration,

$$u = \frac{\pm \arctan \sqrt{-1 + C_1\left(\left(x + \frac{k_1}{k_4}\right)^2 + \left(y - \frac{k5}{k4}\right)^2\right)} + \arctan \frac{y - \frac{k5}{k4}}{x + \frac{k_1}{k_4}}}{k4} + C_2,$$
$$u = \frac{\arcsin \frac{C_1}{\sqrt{\left(\left(x+\frac{k_1}{k_4}\right)^2 + \left(y - \frac{k5}{k4}\right)^2\right)}} + \arctan \frac{y - \frac{k5}{k4}}{x + \frac{k_1}{k_4}}}{k_4}, \tag{16}$$
$$u = \frac{\ln\left(\left(x + \frac{k_5}{k_2}\right)^2 + \left(y + \frac{k_1}{k_2}\right)^2 + 2\left(t + \frac{k_3}{2k_2}\right)\right)}{2k_2}.$$

7. Exact solutions of the affine geometric heat flow

In this section, we carry out the group analysis for the affine case (7) and give exact solutions for a special case.

Now we consider the Lie symmetry of (7). Using the Lie's point symmetry method, we obtain the infinitesimal generator for the symmetry group of (7):

$$X = \eta_1(x, y, t, u)\frac{\partial}{\partial x} + \eta_2(x, y, t, u)\frac{\partial}{\partial y} + \eta_3(x, y, t, u)\frac{\partial}{\partial t} + \eta_4(x, y, t, u)\frac{\partial}{\partial u},$$

where

$$\eta_1 = (3F_5(u) + F_2(u))x + F_4(u)y + F_1(u),$$
$$\eta_2 = F_7(u)x - F_2(u)y + F_3(u),$$
$$\eta_3 = 2F_5(u)t + F_6(u),$$
$$\eta_4 = F_8(u),$$

and $F_i(u)$ $(i = 1, \ldots, 8)$ are eight arbitrary functions.

Here we only consider a special case with $F_i(u) = k_i (i = 1, \ldots, 8)$ where $k_2 \neq 0$, $k_4 = k_5 = 0$, $k_8 = 1$ and other k_i's are arbitrary constants. Then η_1, η_2, η_3 and η_4 become

$$\eta_1^* = k_2x + k_1, \qquad \eta_2^* = k_7x - k_2y + k_3,$$
$$\eta_3^* = k_6, \qquad \eta_4^* = 1.$$

Integration of the characteristic equation

$$\frac{\mathrm{d}x}{k_2x + k_1} = \frac{\mathrm{d}y}{k_7x - k_2y + k_3} = \frac{\mathrm{d}t}{k_6} = \frac{\mathrm{d}u}{1}$$

gives the symmetry invariants

$$\xi = \frac{k_2x + k_1}{\mathrm{e}^{k_2u}k_2},$$
$$\eta = \frac{\mathrm{e}^{k_2u}\left(2k_2^2y - k_2k_7x - 2k_2k_3 + k_7k_1\right)}{2k_2^2},$$
$$\tau = t - k_6u.$$

We now look for a similarity reduction to (7) of the form

$$\tau = v(\xi, \eta).$$

Inserting it into (7) gives

$$-1 = v_\eta^2 v_{\xi\xi} - 2v_\xi v_\eta v_{\xi\eta} + v_\xi^2 v_{\eta\eta}. \tag{17}$$

Now we use Lie group theory to analyze (17). Its Lie algebra of infinitesimal symmetries is spanned by the following seven vector fields:

$$\begin{aligned} &Z_1 = \frac{\partial}{\partial\xi}, \qquad Z_2 = \frac{\partial}{\partial\eta}, \qquad Z_3 = \frac{\partial}{\partial v}, \qquad Z_4 = \xi\frac{\partial}{\partial\eta}, \\ &Z_5 = \eta\frac{\partial}{\partial\xi}, \qquad Z_6 = \xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta}, \qquad Z_7 = 3\xi\frac{\partial}{\partial\xi} + 2v\frac{\partial}{\partial v}. \end{aligned} \tag{18}$$

The adjoint representation for the Lie algebra (18) can be found in table 7.

We now introduce the vectors

$$\begin{aligned} &U_1 = Z_6, \qquad U_2 = Z_6 + Z_3, \qquad U_3 = Z_6 - Z_3, \qquad U_4 = Z_4 - Z_5 + \alpha Z_3, \\ &U_5 = Z_4 + Z_3 + \alpha Z_1, \qquad U_6 = Z_4 - Z_3 + \alpha Z_1, \qquad U_7 = Z_4 + Z_1, \\ &U_8 = Z_4 - Z_1, \qquad U_9 = Z_4, \qquad U_{10} = Z_1, \qquad U_{11} = Z_1 + Z_3, \\ &U_{12} = Z_1 - Z_3, \qquad U_{13} = Z_2, \qquad U_{14} = Z_3, \qquad U_{15} = Z_2 + Z_3, \\ &U_{16} = Z_2 - Z_3, \qquad U_{17} = Z_7 + \alpha Z_6, \qquad U_{18} = Z_7 + Z_4 + \alpha Z_5\left(\alpha < -\tfrac{9}{4}\right), \\ &U_{19} = Z_7 - Z_4 + \alpha Z_5\left(\alpha > \tfrac{9}{4}\right), \qquad U_{20} = Z_7 - \tfrac{3}{2}Z_6 + Z_5, \\ &U_{21} = Z_7 - \tfrac{3}{2}Z_6 - Z_5, \qquad U_{22} = Z_7 + Z_2, \qquad U_{23} = Z_7 - Z_2, \\ &U_{24} = Z_7 - 3Z_6 + Z_2 + Z_1, \qquad U_{25} = Z_7 - 3Z_6 + Z_2 - Z_1, \\ &U_{26} = Z_7 - 3Z_6 + Z_2, \qquad U_{27} = Z_7 - 3Z_6 - Z_2 + Z_1, \\ &U_{28} = Z_7 - 3Z_6 - Z_2 - Z_1, \qquad U_{29} = Z_7 - 3Z_6 - Z_2, \\ &U_{30} = Z_7 - 3Z_6 + Z_1, \qquad U_{31} = Z_7 - 3Z_6 - Z_1. \end{aligned}$$

Theorem 10. *An optimal system of one-dimensional subalgebras of (18) consists of the family* $\{U_i, i = 1, \ldots, 31\}$.

Let $Z = \sum_{i=1}^{7} c_i Z_i$ be a general vector for (18).

Lemma 11. $Q = c_6^2 + 3c_6c_7 + c_4c_5$ *is an invariant.*

Proof. A well-known fact is that the Killing form is invariant under the adjoint action. A straightforward calculation shows that

$$K(Z, Z) = 10\left(c_6^2 + 3c_6c_7 + c_4c_5\right) + 31c_7^2$$

is the Killing form of the Lie algebra (18). Hence $K(Z, Z)$ is invariant under the adjoint action. From lemma 12, we see that Q is an invariant. □

Lemma 12. *The following two functions are invariants:*

$$L = c_7, \qquad S = \operatorname{sign} c_3.$$

After using the optimal system, we obtain 31 nonequivalent one-dimensional subalgebras. With those Lie algebras, one may reduce (17) to ODEs, which are not equivalent essentially [30].

(1) $U_1 = \xi\partial_\xi - \eta\partial_\eta$. For U_1, its invariants are $z = \xi\eta$ and v, the group-invariant solution for (17) is $v = g(z)$, where $g(z)$ satisfies

$$zg'^3 - \tfrac{1}{2} = 0.$$

Solving it gives a solution of (17):

$$v = \frac{3}{2^{\frac{4}{3}}}(\xi\eta)^{\frac{2}{3}} + C_1.$$

(2) $U_2, U_3 = \xi\partial_\xi - \eta\partial_\eta \pm \partial_v$. For U_2 and U_3, the invariants are $z = \xi\eta$ and $v \mp \ln|\xi|$, and the group-invariant solutions for (17) are $v = g(z) \pm \ln|\xi|$, where $g(z)$ satisfies the ODE

$$g'' - 2zg'^3 \mp 3g'^2 + 1 = 0.$$

(3) $U_4 = -\eta\partial_\xi + \xi\partial_\eta + \alpha\partial_v$.
(3.1) $\alpha = 0$. For U_4, its invariants are $z = \xi^2 + \eta^2$ and v, the corresponding group-invariant solution for (17) is $v = g(z)$, then $g(z)$ satisfies the ODE

$$zg'^3 + \tfrac{1}{8} = 0.$$

Solving it, we deduce an exact solution to (17) given by

$$v = -\tfrac{3}{4}(\xi^2 + \eta^2)^{\frac{2}{3}} + C_1.$$

(3.2) $\alpha \neq 0$. In this case, the invariants for U_4 are $z = \xi^2 + \eta^2$ and $v - \alpha \arctan \eta/\xi$, then the group-invariant solution for (17) is $v = g(z) + \alpha \arctan \eta/\xi$, where $g(z)$ satisfies the ODE

$$4\alpha^2 g'' + 8zg'^3 + \frac{6\alpha^2}{z}g' + 1 = 0.$$

(4) $U_5, U_6 = \alpha\partial_\xi + \xi\partial_\eta \pm \partial_v$.
(4.1) $\alpha = 0$. In this case, the invariants are $z = \xi$ and $v \mp \eta/\xi$, and the group-invariant solutions for (17) are given by $v = g(z) \pm \eta/\xi$, where $g(z)$ satisfies the ODE

$$\frac{1}{z^2}g'' + \frac{2}{z^3}g' + 1 = 0.$$

Then, the corresponding solutions to (17) are

$$v = \pm\frac{\eta}{\xi} - \frac{1}{20}\xi^4 + C_1\frac{1}{\xi} + C_2.$$

(4.2) $\alpha \neq 0$. For U_5 and U_6, the invariants are $z = \xi^2 - 2\alpha\eta$ and $v \mp \xi/\alpha$, then the group-invariant solutions for (17) can be represented as $v = g(z) \pm \xi/\alpha$, then $g(z)$ satisfies the ODE

$$4g'' + 8\alpha^2 g'^3 + 1 = 0.$$

(5) $U_7, U_8 = \pm\partial_\xi + \xi\partial_\eta$. The invariants are $z = \xi^2 \mp 2\eta$ and v, and the corresponding group-invariant solutions for (17) are $v = g(z)$, where $g(z)$ satisfies the ODE

$$g'^3 + \tfrac{1}{8} = 0.$$

It gives a solution of (17)

$$v = -\tfrac{1}{2}(\xi^2 \mp 2\eta) + C_1.$$

(6) $U_{11}, U_{12} = \partial_\xi \pm \partial_v$. For U_{11} and U_{12}, the invariants are $z = \eta$ and $v \mp \xi$, and the group-invariant solutions for (17) are $v = g(z) \pm \xi$, where $g(z)$ satisfies the ODE

$$g'' + 1 = 0.$$

The corresponding solutions to (17) are given by

$$v = \pm\xi - \tfrac{1}{2}\eta^2 + C_1\eta + C_2.$$

(7) $U_{15}, U_{16} = \partial_\eta \pm \partial_v$. For U_{15} and U_{16}, the invariants are $z = \xi$ and $v \mp \eta$, the group-invariant solutions for (17) are $v = g(z) \pm \eta$, where $g(z)$ satisfies

$$g'' + 1 = 0.$$

It gives a solution of (17):

$$v = \pm\eta - \tfrac{1}{2}\xi^2 + C_1\xi + C_2.$$

(8) $U_{17} = (3+\alpha)\xi\partial_\xi - \alpha\eta\partial_\eta + 2v\partial_v$.
(8.1) $\alpha = 0$. In this case, its invariants are $z = \eta$ and $v\xi^{-2/3}$, the group-invariant solution for (17) is given by $v = \xi^{2/3}g(z)$, where $g(z)$ satisfies the ODE

$$4g^2g'' - 10gg'^2 + 9 = 0.$$

(8.2) $\alpha \neq 0$. For U_{17}, its invariants are $z = \eta^{1+3/\alpha}\xi$ and $v\eta^{2/\alpha}$, and the corresponding group-invariant solution for (17) is given by $v = \eta^{-2/\alpha}g(z)$, where $g(z)$ satisfies the ODE

$$4g^2g'' - (2\alpha^2 + 9\alpha + 9)zg'^3 + 2(3\alpha + 4)gg'^2 + \alpha^2 = 0.$$

(9) $U_{18} = (3\xi + \alpha\eta)\partial_\xi + \xi\partial_\eta + 2v\partial_v \left(\alpha < -\tfrac{9}{4}\right)$. For U_{18}, its invariants are $z = (\xi^2 - 3\xi\eta - \alpha\eta^2)/(4\xi^2 - 12\xi\eta + 9\eta^2)$ and $v/(2\xi - 3\eta)^{4/3}$, the group-invariant solution for (17) is given by $v = (2\xi - 3\eta)^{4/3}g(z)$, where $g(z)$ satisfies the ODE

$$(1-4z)g^2g'' - \frac{1}{4}(1-4z)gg'^2 - 2g^2g' + \frac{9}{16(9+4\alpha)} = 0.$$

(10) $U_{19} = (3\xi + \alpha\eta)\partial_\xi - \xi\partial_\eta + 2v\partial_v \left(\alpha > \tfrac{9}{4}\right)$. For U_{19}, similar as U_{18}, its invariants are $z = (\xi^2 + 3\xi\eta + \alpha\eta^2)/(4\xi^2 + 12\xi\eta + 9\eta^2)$ and $v/(2\xi + 3\eta)^{4/3}$, the group-invariant solution for (17) takes the form $v = (2\xi + 3\eta)^{4/3}g(z)$, where $g(z)$ satisfies the ODE

$$(1-4z)g^2g'' - \frac{1}{4}(1-4z)gg'^2 - 2g^2g' + \frac{9}{16(9-4\alpha)} = 0.$$

(11) $U_{20}, U_{21} = \left(\tfrac{3}{2}\xi \pm \eta\right)\partial_\xi + \tfrac{3}{2}\eta\partial_\eta + 2v\partial_v$. For U_{20} and U_{21}, the invariants are $z = (2/3)\ln|\eta| \mp \xi/\eta$ and $v/\eta^{4/3}$, the group-invariant solutions for (17) are $v = \eta^{4/3}g(z)$. Then $g(z)$ satisfies

$$16g^2g'' + 6g'^3 - 4gg'^2 + 9 = 0.$$

(12) $U_{22}, U_{23} = 3\xi\partial_\xi \pm \partial_\eta + 2v\partial_v$. For U_{22} and U_{23}, the invariants are $z = \ln|\xi| \mp 3\eta$ and $v\xi^{-2/3}$, the group-invariant solution for (17) is $v = \xi^{2/3}g(z)$, where g fulfils the ODE

$$4g^2g'' - 9g'^3 - 10gg'^2 + 1 = 0. \tag{19}$$

(13) $U_{24}, U_{25} = \pm\partial_\xi + (3\eta + 1)\partial_\eta + 2v\partial_v$. For U_{24} and U_{25}, the invariants are $z = (1/3)\ln|3\eta + 1| \mp \xi$ and $v(3\eta + 1)^{-2/3}$, the group-invariant solution for (17) is given by $v = (3\eta + 1)^{2/3} g(z)$, where $g(z)$ satisfies the ODE

$$4g^2 g'' - 3g'^3 - 10gg'^2 + 1 = 0. \tag{20}$$

(14) $U_{26} = (3\eta + 1)\partial_\eta + 2v\partial_v$. For U_{26}, the invariants are $z = \xi$ and $v(3\eta + 1)^{-2/3}$, the group-invariant solution for (17) is given by $v = (3\eta + 1)^{2/3} g(z)$, where $g(z)$ satisfies

$$4g^2 g'' - 10gg'^2 + 1 = 0. \tag{21}$$

(15) $U_{27}, U_{28} = \pm\partial_\xi + (3\eta - 1)\partial_\eta + 2v\partial_v$. For U_{27} and U_{28}, the invariants are $z = (1/3)\ln|3\eta - 1| \mp \xi$ and $v(3\eta - 1)^{-2/3}$, the group-invariant solution for (17) is $v = (3\eta - 1)^{2/3} g(z)$ with $g(z)$ satisfying (20).

(16) $U_{29} = (3\eta - 1)\partial_\eta + 2v\partial_v$. The invariants for U_{29} are $z = \xi$ and $v(3\eta - 1)^{-2/3}$, the group-invariant solution for (17) is given by $v = (3\eta - 1)^{2/3} g(z)$, where $g(z)$ satisfies equation (21).

(17) $U_{30}, U_{31} = \pm\partial_\xi + 3\eta\partial_\eta + 2v\partial_v$. For U_{30} and U_{31}, the invariants are $z = \ln|\eta| \mp 3\xi$ and $v\eta^{-2/3}$, the group-invariant solution for (17) is $v = \eta^{2/3} g(z)$, with $g(z)$ satisfying (19).

Now the symmetry group analysis for (17) is accomplished, since we have reduced the affine geometric heat flow (7) to (17) for the special case defined before. Then combining the results and conclusions obtained above, the group-invariant solutions of the affine geometric heat flow in the case $F_i(u) = k_i (i = 1, \ldots, 8)$, where $k_2 \neq 0, k_4 = k_5 = 0, k_8 = 1$ and other k_i's are arbitrary constants, can be expressed as

$$t - k_6 u = v\left(\frac{k_2 x + k_1}{e^{k_2 u} k_2}, \frac{e^{k_2 u}\left(2k_2^2 y - k_2 k_7 x - 2k_2 k_3 + k_7 k_1\right)}{2k_2^2}\right),$$

where v satisfies (17).

8. Concluding remarks

We have systematically derived the Lie point symmetries of the geometric heat flow (6). The basic similarity reductions are performed when the arbitrary functions in the infinitesimal transformations are confined to constants. Reduced equations and exact solutions associated with the symmetries are obtained.

Lie symmetries for the affine geometric heat flow (7) are also determined and its corresponding group-invariant solutions are also derived for a special case.

It remains open to reduce equations (6) and (7) when the functions $F_i(u)$ are not constants.

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Appendix

We put tables A1 and A2 cited in sections 4 and 5, respectively, in the appendix.

Table A1. Reduced equation for (10).

No	Generators	Invariants	Ansatz	Reduced equation
1	X_4	$(\frac{\xi}{\eta}, v)$	$v = g(z)$	$4zg'' + 4z^2(z+2)g'^3 + (z+6)g' = 0$
2	$X_4 + \alpha X_3$	$(\frac{\xi}{\eta}, v - \alpha \ln\|\eta\|)$	$v = g(z) + \alpha \ln\|\eta\|$	$4zg'' + 4z^2(z+2)g'^3 - 4\alpha z^2 g'^2 + (z+6)g' - \alpha = 0$
4	$X_3 + X_5$	$(\xi, v - \eta)$	$v = g(z) + \eta$	$4zg'' + 8z^2g'^3 - 4z^2g'^2 + 6g' - 1 = 0$
5	$X_3 - X_5$	$(\xi, v + \eta)$	$v = g(z) - \eta$	$4zg'' + 8z^2g'^3 + 4z^2g'^2 + 6g' + 1 = 0$
6	X_1	$(\eta, \xi \sin^2 v)$	$\xi \sin^2 v = g(z)$	$g' = 0$
7	X_5	(ξ, v)	$v = g(z)$	$4zg'' + 8z^2g'^3 + 6g' = 0$
8	$X_5 + X_1$	$(\xi \sin^2 v, \sqrt{\xi}\cos v - \eta)$	$\sqrt{\xi}\cos v - \eta = g(z)$	$4zg'' - 4zg'^2 + 2g' - 1 = 0$

Table A2. Reduced equation for (11).

No	Generators	Invariants	Ansatz	Reduced equation
1	Y_4	$(\xi^2+\eta^2, v)$	$v = g(z)$	$g' + \frac{1}{2} = 0$
2	$Y_4 + Y_3$	$(\xi^2+\eta^2, v - \arctan\frac{\eta}{\xi})$	$v = g(z) + \arctan\frac{\eta}{\xi}$	$4zg'' + 8z^2g'^3 + 4z^2g'^2 + 6g' + 1 = 0$
3	$Y_4 - Y_3$	$(\xi^2+\eta^2, v + \arctan\frac{\eta}{\xi})$	$v = g(z) - \arctan\frac{\eta}{\xi}$	$4zg'' + 8z^2g'^3 + 4z^2g'^2 + 6g' + 1 = 0$
4	$Y_4 + \alpha Y_5$	$(\ln(\xi^2+\eta^2) - 2\alpha\arctan\frac{\eta}{\xi}, \frac{v}{\xi^2+\eta^2})$	$v = (\xi^2+\eta^2)g(z)$	$4\alpha^2g^2g'' + 2(\alpha^2+1)g'^3$ $+ (\alpha^2+1+6g-2\alpha^2g)g'^2 + 2(3g+1)gg'$ $+ g^2(2g+1) = 0$
5	Y_5	$(\frac{\eta}{\xi}, \frac{v}{\eta^2})$	$v = \eta^2 g(z)$	$4z^4g^2g'' + z^2(z^2+1-2z^2g)g'^2$ $+ 4z(2z^2g+1)gg' + 4g^2 = 0$
8	$Y_1 + Y_3$	$(\eta, v - \xi)$	$v = g(z) + \xi$	$g'' + g'^2 + 1 = 0$

References

[1] Angenent S 1990 Parabolic equations for curves on surfaces: Part I. Curves with p-integrable curvature *Ann. Math.* **132** 451–83

[2] Angenent S 1991 On the formation of singularities in the curve shortening flow *J. Diff. Geom.* **33** 601–34

[3] Gage M and Hamilton R S 1986 The heat equation shrinking convex plane curves *J. Diff. Geom.* **23** 69–96

[4] Grayson M 1989 Shortening embedded curves *Ann. Math.* **129** 71–111

[5] Chow B, Lu P and Ni L 2006 *Hamilton's Ricci Flow (Graduate Studies in Mathematics* vol 77*)* (Providence, RI/New York: American Mathematical Society/Science Press)

[6] Kimia B B, Tannenbaum A and Zucker S W 1992 On the evolution of curves via a function of curvature: I. The classical case *J. Math. Anal. Appl.* **163** 438–58

[7] Sapiro G 2001 *Geometric Partial Differential Equations and Image Analysis* (New York: Cambridge University Press)

[8] Cao F 2003 *Geometric Curve Evolution and Image Processing* (Berlin: Springer)

[9] Kimia B B, Tannenbaum A and Zucker S W 1995 Shapes, shocks, and deformations: I. The components of two-dimensional shape and the reaction–diffusion space *Int. J. Comput. Vis.* **15** 189–224

[10] Kimia B B and Siddiqi 1996 Geometric heat equation and nonlinear diffusion of shapes and images *Comput. Vis. Image Understand.* **64** 305–22

[11] Dolcetta I C, Vita S F and March R 2002 Area-preserving curve-shortening flows: from phase separation to image processing *Interfaces Free Bound.* **4** 325–43

[12] Angenent S, Pichon E and Tannenbaum A 2006 Mathematical methods in medical image processing *Bull. AMS* **43** 365–96

[13] Deckelnick K, Dziuk G and Elliott C M 2005 Computation of geometric partial differential equations and mean curvature flow *Acta Numer.* **14** 139–232

[14] Bakas I and Sourdis C 2007 Dirichlet sigma models and mean curvature *Preprint* hep-th/0704.3985v1

[15] Gage M 1983 An isoperimetric inequality with applications to curve shortening *Duke Math. J.* **50** 1225–9

[16] Gage M 1984 Curve shortening makes convex curves circular *Invent. Math.* **76** 357–64

[17] Grayson M 1987 The heat equation shrinks embedded plane curves to round points *J. Diff. Geom.* **26** 285–314

[18] Alvarez L, Guichard F, Lions P L and Morel J M 1993 Axioms and fundamental equations of image processing *Arch. Ration. Mech. Anal.* **123** 199–257

[19] Sapiro G and Tannenbaum A 1993 Affine invariant scale space *Int. J. Comput. Vis.* **11** 25–44

[20] Sapiro G and Tannenbaum A 1993 On invariant curve evolution and image analysis *Indiana Univ. Math. J.* **42** 985–1011

[21] Angenent S, Sapiro G and Tannenbaum A 1998 On the affine heat equation for non-convex curves *J. Am. Math. Soc.* **11** 601–34

[22] Chou K S and Zhu X P 2001 *The Curve Shortening Problem* (London/Boca Raton, FL: Chapman and Hall/CRC Press)

[23] Chou K S and Li G X 2002 Optimal systems and invariant solutions for the curve shortening problem *Commun. Anal. Geom.* **10** 241–74

[24] Osher S J and Sethian J A 1988 Fronts propagating with curvature dependent speed: algorithms based on the Hamilton–Jacobi formulation *J. Comput. Phys.* **79** 12–49

[25] Sethian J A 1996 *Level Set Methods: Evolving Interfaces in Geometry, Fluid Mechanics, Computer Vision and Materials Sciences* (Cambridge: Cambridge University Press)

[26] Alvarez L, Lions P L and Morel J M 1992 Image selective smoothing and edge detection by nonlinear diffusion *SIAM J. Numer. Anal.* **29** 845–66

[27] Abresch U and Langer J 1986 The normalized curve shortening flow and homothetic solutions *J. Diff. Geom.* **23** 175–96

[28] Nien C H and Tsai D H 2006 Convex curves moving translationally in the plane *J. Diff. Eqns* **225** 605–23

[29] Ibragimov N H 1985 *Transformation Groups Applied to Mathematical Physics* (Dordrecht: Reidel)

[30] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (New York: Springer)

[31] Ovsiannikov L V 1982 *Group Analysis of Differential Equations* (New York: Academic)

[32] Chou K S and Qu C Z 2004 Optimal systems and group classification of (1+2)-dimensional heat equation *Acta Appl. Math.* **83** 257–87